Math 6261 23-02-17

Recall that

Thm (Borel-Cantelli Lemma) Suppose $\sum_{n=1}^{\infty} P(A_n) < \infty$.

Then $P(A_n \text{ i.o.}) = 0$.

As an application of the Borel-Cantelli Lemma, we have the following version of strong law of large numbers.

Thm 2.11. Let $X_1, X_2, \dots,$ be i.i.d. with $EX_i = \mu$ and $EX_i^4 < \infty$.

Then $\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu$ a.s.

Pf. Replacing X_i by $X_i - \mu$, we may assume that $EX_i = 0$.

Set $S_n = X_1 + \cdots + X_n$. We will show that $\lim_{n\to\infty} \frac{S_n}{n} = 0$ a.s

Notice that

$$E S_n^4 = E\left(\sum_{1\le i,j,k,l\le n} X_i X_j X_k X_l\right)$$

$$= \sum_{1\le i,j,k,l\le n} E(X_i X_j X_k X_l)$$

Terms in the above sums of the form

$$E(X_i^3 X_j), \quad E(X_i^2 X_j X_k), \quad E(X_i X_j X_k X_l)$$

(if i, j, k, l are distinct)

are all 0. The remaining terms are of the form $E(X_i^2 X_j^2)$ $(i \ne j)$ and $E(X_i^4)$.

Hence $\quad E S_n^4 = \overset{\binom{n}{2}\cdot\binom{4}{2}}{} E(X_1^2 X_2^2) + \overset{n}{} E(X_1^4)$

$$= 3\,n(n-1)\; E(X_1^2 X_2^2) + n\, E(X_1^4)$$

$$\leq 3\,n(n-1)\; E(X_1^4)^{\frac{1}{2}}\, E(X_2^4)^{\frac{1}{2}} + n\, E(X_1^4)$$

$$= (3n^2 - 2n)\; E X_1^4$$

Hence $\quad E S_n^4 \leq C n^2$.

Let $\varepsilon > 0$. Then by Chebyshev inequality,

$$P(\,|S_n| > n\varepsilon\,) \leq \frac{E\, S_n^4}{n^4 \varepsilon^4} \leq \frac{C}{n^2 \varepsilon^4}$$

So $\quad \sum_{n=1}^{\infty} P(\,|S_n| > n\varepsilon\,) < \infty$.

It follows that $\quad P(\,|S_n| > n\varepsilon \text{ i.o.}\,) = 0$. Hence $\overline{\lim_{n\to\infty}} \frac{|S_n|}{n} \leq \varepsilon$ a.s.

Since ε is arbitrary, $\frac{S_n}{n} \to 0$ almost surely. $\square$

Thm 2.12. (The second Borel-Cantelli lemma).

If the events A_n are independent with $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $\quad P(\,A_n \text{ i.o.}) = 1$.

pf. $\quad$ Let $M < N$. By independence and $1 - x < e^{-x}$

$$P\left(\bigcap_{n=M}^{N} A_n^c\right) = \prod_{n=M}^{N}\left(1-P(A_n)\right)$$
$$\leq \prod_{n=M}^{N} e^{-P(A_n)}$$
$$= e^{-\sum_{n=M}^{N} P(A_n)} \to 0 \quad \text{as } N\to\infty$$

Hence $P\left(\bigcap_{n=M}^{\infty} A_n^c\right) = 0 \Rightarrow P\left(\bigcup_{n=M}^{\infty} A_n\right)=1 \Rightarrow P(A_n \text{ i.o.})=1.$ $\square$

<u>Cor. 2.13.</u> If $X_1, X_2, \cdots,$ are i.i.d with $E|X_i| = \infty$, then

$$P(|X_n| \geq n \text{ i.o.}) = 1,$$

and

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} \in (-\infty, \infty)\right) = 0.$$

Pf. $E|X_1| = \int_0^\infty P(|X_1| > t)\, dt$

$$\leq \sum_{n=1}^{\infty} \int_{n-1}^{n} P(|X_1| > t)\, dt$$
$$\leq \sum_{n=1}^{\infty} P(|X_1| > n-1)$$
$$= \sum_{n=1}^{\infty} P(|X_{n-1}| > n-1)$$

By the second BC lemma, $P(|X_n| > n \text{ i.o.}) = 1$.

To prove the second claim, observe that

$$\frac{S_n}{n} - \frac{S_{n+1}}{n+1} = \frac{S_n}{n} - \frac{S_n + X_{n+1}}{n+1} = \frac{S_n}{n(n+1)} - \frac{X_{n+1}}{n+1}.$$

Write $C = \left\{ \omega : \lim_{n\to\infty} \frac{S_n}{n} \in (-\infty, \infty) \right\}$.

On $C \cap \{ \omega : |X_n| > n \text{ i.o.} \}$, $\frac{S_n}{n(n+1)} \to 0$ so

$$\left| \frac{S_n}{n} - \frac{S_{n+1}}{n+1} \right| \geqslant \frac{1}{2} \text{ i.o.}$$

contradicting the fact that $\omega \in C$. Hence $C \cap \{ \omega : |X_n| > n \text{ i.o.} \} = \phi$

which implies $P(C) = 0$.

Remark: The above result shows that the condition

$$E|X_i| < \infty$$

in the strong law of large numbers is necessary.

§2.4 Strong law of large numbers.

Thm 2.14. Let $X_1, \cdots, X_n, \cdots$ be pairwise independent, identically distributed r.v.'s with $E|X_i| < \infty$. Let $\mu = EX_i$.
Then
$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu \quad \text{a.s.}$$

Below we will follow Etemadi's proof.

Lem A Let $Y_k = X_k \mathbb{1}_{(|X_k| \le k)}$ and
$$T_n = Y_1 + \cdots + Y_n.$$
Then it suffices to show that $\frac{T_n}{n} \to \mu$ a.s.

Pf.
$$\sum_{k=1}^{\infty} P(|X_k| > k)$$
$$= \sum_{k=1}^{\infty} P(|X_1| > k) \le \int_0^{\infty} P(|X_1| > t)\,dt = E|X_1| < \infty.$$
By the Borel-Cantelli lemma,
$$P(|X_k| > k \text{ i.o.}) = 0.$$
Equivalently,
$$P(X_k \ne Y_k \text{ i.o.}) = 0.$$

This shows that $|S_n(\omega) - T_n(\omega)| \le R(\omega) < \infty$ a.s. for all n.

Hence $\lim_{n\to\infty} \frac{T_n}{n} = \mu$ a.s $\Rightarrow \lim_{n\to\infty} \frac{S_n}{n} = \mu$ a.s. ∎

Lem B. $$\sum_{k=1}^{\infty} \frac{Var(Y_k)}{k^2} < 4E|X_1| < \infty.$$

pf. $$Var(Y_k) \le E(Y_k^2) = \int_0^{\infty} P(|Y_k|^2 > t)\, dt$$

(Let $t = y^2$)

$$= \int_0^{\infty} 2y\, P(|Y_k| > y)\, dy$$

$$= \int_0^{k} 2y\, P(|Y_k| > y)\, dy$$

$$\le \int_0^{k} 2y\, P(|X_k| > y)\, dy = \int_0^{k} 2y\, P(|X_1| > y)\, dy.$$

Hence

$$\sum_{k=1}^{\infty} \frac{Var(Y_k)}{k^2} \le \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{k} 2y\, P(|X_1| > y)\, dy$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\infty} 2y \cdot \mathbb{1}_{(y<k)}\, P(|X_1| > y)\, dy$$

$$= \int_0^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \mathbb{1}_{(y<k)} \right) \cdot 2y\, P(|X_1| > y)\, dy$$

$$\le \int_0^{\infty} \left(\sum_{k=1}^{\infty} \frac{2}{k(k+1)} \cdot \mathbb{1}_{(y<k)} \right) \cdot 2y\, P(|X_1| > y)\, dy$$

$$\le \int_0^{\infty} \left(\sum_{k=[y]+1}^{\infty} 2 \cdot \left(\frac{1}{k} - \frac{1}{k+1} \right) \right) \cdot (2y)\, P(|X_1| > y)\, dy$$

$$\le \int_0^\infty \frac{2}{[y]+1} \cdot 2y \cdot P(|X_1|>y)\,dy$$

$$\le 4 \int_0^\infty P(|X_1|>y)\,dy$$

$$= 4\ E|X_1|.$$

$\blacksquare$

<u>Pf of the strong law of large numbers</u>:

Since both X_n^+, X_n^- satisfy the assumptions of the theorem and $X_n = X_n^+ - X_n^-$, so we can WLOG assume that $X_n \geq 0$.

Now we will first prove the result for a subsequence, and then use monotonicity to control the values in between.

Let $\alpha > 1$ and $k(n) = [\alpha^n]$, where $[x]$ denotes the integral part of x.

For $\varepsilon > 0$, we have

$$\sum_{n=1}^\infty P\left(|T_{k(n)} - E T_{k(n)}| > \varepsilon \cdot k(n) \right)$$

(Chebyshev)

$$\le \varepsilon^{-2} \cdot \sum_{n=1}^\infty \frac{Var(T_{k(n)})}{k(n)^2}$$

$$= \varepsilon^{-2} \sum_{n=1}^\infty \frac{1}{k(n)^2} \cdot \sum_{m=1}^{k(n)} Var(Y_m)$$

(in which we use the pairwise independently assumption)

$$= \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{k(n)^2} \cdot \mathbb{1}_{\{m \leq k(n)\}} \operatorname{Var}(Y_m)$$

$$= \varepsilon^{-2} \sum_{m=1}^{\infty} \operatorname{Var}(Y_m) \cdot \sum_{n:\, k(n) \geq m} \frac{1}{k(n)^2} \cdot \quad (1)$$

Notice that

$$\sum_{n:\, [\alpha^n] \geq m} \frac{1}{[\alpha^n]^2} \leq \sum_{n:\, \alpha^n \geq m} 4 \cdot \alpha^{-2n}$$

$$= 4 \cdot \sum_{n = \lceil \frac{\log m}{\log \alpha} \rceil} \alpha^{-2n}$$

$$= 4 \cdot \alpha^{-2 \lceil \frac{\log m}{\log \alpha} \rceil} \cdot \frac{1}{1-\alpha^{-2}}$$

$$\leq 4 \cdot \alpha^{-2 (\log_\alpha m)} \frac{1}{1-\alpha^{-2}}$$

$$= 4 \cdot \frac{1}{m^2} \cdot \frac{1}{1-\alpha^{-2}} .$$

So by (1),

$$\sum_{n=1}^{\infty} P\left(\left| T_{k(n)} - E\, T_{k(n)} \right| > \varepsilon\, k(n) \right)$$

$$\leq \varepsilon^{-2} \cdot \sum_{m=1}^{\infty} \frac{\operatorname{Var}(Y_m)}{m^2} \cdot \frac{4}{1-\alpha^{-2}}$$

$< \infty$ (by Lem B).

By the Borel-Cantelli lemma,

$$\overline{\lim_{n \to \infty}} \frac{\left| T_{k(n)} - E\, T_{k(n)} \right|}{k(n)} \leq \varepsilon \quad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrarily given, we have

$$\lim_{n\to\infty} \frac{T_{k(n)} - E\,T_{k(n)}}{k(n)} = 0 \qquad (2)$$

However $E\,Y_k = E\left(X_k \mathbb{1}_{(X_k \le k)}\right)$

$$= E\left(X_1 \cdot \mathbb{1}_{(X_1 \le k)}\right) \to E X_1 = \mu \text{ as } k\to\infty$$

(by the monotone convergence Thm).

It follows that $\frac{E\,T_{k(n)}}{k(n)} \to \mu$ as $n\to\infty$.

So by (2),

$$\lim_{n\to\infty} \frac{T_{k(n)}}{k(n)} = \mu.$$

Now for a given $m \in \mathbb{N}$, let n s.t

$$k(n) < m < k(n+1).$$

Then

$$\frac{T_{k(n)}}{k(n+1)} \le \frac{T_m}{m} \le \frac{T_{k(n+1)}}{k(n)}$$

Since $\frac{k(n+1)}{k(n)} \to \alpha$ as $n\to\infty$, it follows that

$$\frac{\mu}{\alpha} \le \varliminf_{m\to\infty} \frac{T_m}{m} \le \varlimsup_{m\to\infty} \frac{T_m}{m} \le \alpha\cdot\mu \quad \text{almost surely.}$$

But since $\alpha > 1$ is arbitrary, we get

$$\lim_{m\to\infty} \frac{T_m}{m} = \mu \quad a.s.$$

$\blacksquare$.

The next result shows that SLLN holds whenever EX_i exists.

Thm 2.15. Let $X_1, X_2, \cdots$, be i.i.d with $EX_i^+ = \infty$ and $EX_i^- < \infty$. Then

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty \quad a.s.$$

Pf. Let $M > 0$. Set $X_i^M = \min\{X_i, M\}$.

Then X_i^M are i.i.d. with $E|X_i^M| < \infty$.

By the SLLN,

$$\frac{X_1+\cdots+X_n}{n} \geq \frac{X_1^M + \cdots + X_n^M}{n} \to EX_i^M \quad as \quad n\to\infty.$$

By the monotone convergence Thm,

Since $(X_i^M)^+ \nearrow X_i^+$, $\quad E(X_i^M)^+ \to EX_i^+$ as $M\to\infty$.

But $E(X_i^M)^- = EX_i^-$,

So we have

$$E X_i^M = E(X_i^M)^+ - E(X_i^M)^-$$

$$\longrightarrow E X_i^+ - E X_i^- = E X_i \quad \text{as } M \to \infty.$$

$= \infty$

It follows that

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty \quad \text{a.s.} \qquad \blacksquare$$